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Estimates for the Lorentz Degree of Polynomials

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DEDICATED TO PROFESSOR GEORGE G. LORENTZ ON HIS 80TH BIRTHDAY

In 1916 S. N. Bernstein observed that every polynomial p having no zeros in $(-1, 1)$ can be written in the form $\sum_{j=0}^{d} a_j(1-x)^j(1+x)^{d-j}$ with all $a_j \geqslant 0$ or all $a_j \leqslant 0$. The smallest natural number d for which such a representation holds is called the Lorentz degree of p and it is denoted by $d(p)$. The Lorentz degree $d(p)$ can be much larger than the ordinary degree $\deg(p)$. In this paper lower bounds are given for the Lorentz degree of certain special polynomials which show the sharpness of some earlier results of T. Erdélyi and J. Szabados. As a by-product, we give a short proof of Markov and Bernstein type inequalities for the derivatives of polynomials of the above form, established by G. G. Lorentz in 1963. The Lorentz degree of trigonometric polynomials is also introduced and some analogues of our algebraic results are established.

1. INTRODUCTION AND NOTATIONS

Denote by Π_n the set of all real algebraic polynomials of degree at most n. A Lorentz representation of a polynomial p is a representation

$$p(x) = \sum_{j=0}^{d} a_j(1-x)^j(1+x)^{d-j}. \tag{1}$$

One of the interesting properties of this representation is that every polynomial p, positive in $(-1, 1)$, possesses a representation (1) with all $a_j \geqslant 0$. This was observed by S. N. Bernstein [1] in 1916. Another simple observation is that every polynomial p with integer coefficients has a representation (1) with all a_j integers and in which d is equal to the degree of p. The representation (1) turned out to be useful in different parts of approximation theory [4, 7, 8–10] and helped indirectly to inspire others [11, 12]. The classes

$$P_d(a, b) = \left\{p : p(x) = \sum_{j=0}^{d} a_j(b-x)^j(x-a)^{d-j} \text{ with all } a_j \geqslant 0 \text{ or all } a_j \leqslant 0\right\}$$

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were introduced by G. G. Lorentz [9], who proved several interesting approximation theoretic properties about them such as sharp Markov and Bernstein type inequalities. The classes $P_d(-1, 1)$ were studied in [4] as well. For a $p \in \Pi_n$ having no zeros in (a, b) we defined its Lorentz degree $d_{[a,b]}(p)$ as the minimal d for which $p \in P_d(a, b)$ (this is a well-defined natural number by the already mentioned observation of S. N. Bernstein). If $p \not\equiv 0$ vanishes in (a, b), then obviously there is no d such that $p \in P_d(a, b)$; in this case we say $d_{[a,b]} = \infty$. Since we will work in $[-1, 1]$ mostly, we will use the abbrevation

$$d(p) := d_{[-1,1]}(p). \tag{2}$$

Observe that [4, p. 108]

$$d_{[\alpha,\beta]}(p) \leqslant d_{[a,b]}(p) \qquad \text{if} \quad [\alpha, \beta] \subset [a, b]. \tag{3}$$

The magnitude of $d(p)$ was examined in [4]. To formulate our main theorem from [4] we need some notations. Let $\varphi(x)$ be a positive continuous function on $(-1, 1)$, and let

$$D(\varphi) = \{z = x + iy \colon |y| < \varphi(x), |x| < 1\}$$

denote the domain of the complex plane determined by it. We introduce

$$L_n(\varphi) = \{p \colon p \in \Pi_n, p(z) \neq 0 \text{ in } D(\varphi)\}$$

and

$$d_n(\varphi) = \sup_{p \in L_n(\varphi)} d(p).$$

In [4, Theorem 3] we proved the following

THEOREM A. *If*

$$1 \geqslant \varepsilon := \inf_{|x|<1} \frac{\varphi(x)}{\sqrt{1-x^2}} > 0, \tag{4}$$

then there exist absolute constants $c_1, c_2 > 0$ such that

$$\frac{c_1 n}{2\varepsilon^{4/3}} \leqslant c_1 n \max\left\{\frac{1-a^2}{\varepsilon^2}, \frac{1}{\varepsilon(\varepsilon + \sqrt{1-a^2})}\right\} \leqslant d_n(\varphi) \leqslant \frac{c_2 n}{\varepsilon^2},$$

where a ($|a| < 1$) is a point where the infimum in (4) is attained.

We remark that if $\varepsilon \geqslant 1$ in (4), then Theorem 5 from [4] shows that $d_n(\varphi) = n$. In this paper we show that the lower bound $c_1 n/(2\varepsilon^{4/3})$ in Theorem A can be improved to $c_3 n/\varepsilon^2$ which matches the upper bound.

Let C be the open unit disk of the complex plane. In [4, Theorem 5] we also proved the following.

THEOREM B. (i) *If* $D(\varphi) \supseteq C$, *then*

$$d_n(\varphi) = n \qquad \text{for all} \quad n = 1, 2, \ldots. \tag{5}$$

(ii) *If* (5) *holds for some* $n \geq 2$, *then* $D(\varphi) \supseteq C$.

Thus the functions φ for which $d_n(\varphi) = n$ are completely characterized. To characterize those polynomials p for which $d(p) = \deg(p)$ (where $\deg(p)$ denotes the ordinary degree of p) seems to be much more difficult. Here we show that a polynomial p satisfying $d(p) = \deg(p)$ can have arbitrarily many prescribed zeros outside $(-1, 1)$.

As a by-product, in Section 6 we give a short proof of Markov and Bernstein type inequalities for the derivatives of polynomials from $P_d(-1, 1)$, established by G. G. Lorentz [9] in 1963.

The Lorentz degree of trigonometric polynomials was introduced in [5]. Some trigonometric analogues of our algebraic results are obtained in Section 7.

Finally, we present some conjectures in Section 8.

2. NEW RESULTS

We improve Theorem A by

THEOREM 1. *If*

$$1 \geq \varepsilon := \inf_{|x| < 1} \frac{\varphi(x)}{\sqrt{1 - x^2}} > 0,$$

then

$$\frac{c_3 n}{\varepsilon^2} \leq d_n(\varphi) \leq \frac{c_2 n}{\varepsilon^2},$$

where $0 < c_3 < c_2$ *are absolute constants.*

Our next theorem shows that a polynomial p satisfying $d(p) = \deg(p)$ can have arbitrarily many prescribed zeros outside $(-1, 1)$.

THEOREM 2. *If* $q \in \Pi_k \backslash \Pi_{k-1}$ *has no zeros in* $(-1, 1)$, $z \in \mathbb{C}$, $|z| > 1$, *and* $p(x) = ((x - z)(x - \bar{z}))^m q(x)$, *then* $d(p) = \deg(p) = k + 2m$ *if* m *is large enough.*

3. Lemmas for Theorem 3

To prove Theorem 3 we need some lemmas.

Lemma 1. *Let $p \in P_d(-1,1)$, $1 \leqslant n \leqslant d$ and $0 \leqslant a < 1$. Then*

$$\left| p\left(a + i\frac{\sqrt{1-a^2}\sqrt{n}}{2\sqrt{d}}\right)\right| \leqslant c_4^n \left| p\left(a - \frac{n}{4d}\right)\right|,$$

where c_4 is an absolute constant (i is the imaginary unit).

Lemma 2. *We have*

$$p'(b) \leqslant dp(b) \qquad (0 \leqslant b \leqslant 1)$$

for every polynomial $p \in P_d(-1,1)$, positive in $(-1,1)$.

Lemma 3. *Let*

$$p(x) = ((x-a)^2 + \varepsilon^2(1-a^2))^n \qquad (-1 < a < 1, 0 < \varepsilon \leqslant 1).$$

Then $d(p) \geqslant c_5 n/\varepsilon^2$ with some absolute constant $c_5 > 0$.

4. Proofs of the Lemmas for Theorem 1

Proof of Lemma 1. Let $p \in P_d(-1,1)$; that is, let $p(x) = \sum_{j=0}^{d} a_j(1-x)^j(1+x)^{d-j}$ with all $a_j \geqslant 0$ or all $a_j \leqslant 0$. Hence it is sufficient to prove the inequality of the lemma only for the polynomials

$$q_{d,j}(x) = (1-x)^j(1+x)^{d-j} \qquad (j = 0, 1, ..., d). \tag{6}$$

We have

$$\begin{aligned}
&\left| q_{d,j}\left(a + i\frac{\sqrt{1-a^2}\sqrt{n}}{2\sqrt{d}}\right)\right| \\
&= \left((1-a)^2 + \frac{n}{4d}(1-a^2)\right)^{j/2} \left((1+a)^2 + \frac{n}{4d}(1-a^2)\right)^{(d-j)/2} \\
&\leqslant \left((1-a) + \frac{n}{4d}\right)^j \left((1+a) + \frac{n}{4d}\right)^{d-j} \\
&= \left(1 - \left(a - \frac{n}{4d}\right)\right)^j \left(1 + \left(a - \frac{n}{4d}\right)\right)^{d-j} \left(\frac{1+a+n/(4d)}{1+a-n/(4d)}\right)^{d-j} \\
&\leqslant q_{d,j}\left(a - \frac{n}{4d}\right)\left(1 - \frac{n}{2d}\right)^{-d} \leqslant c_4^n q_{d,j}\left(a - \frac{n}{4d}\right);
\end{aligned}$$

thus the lemma is proved. ∎

Proof of Lemma 2. Since $p \in P_d(-1, 1)$, is positive in $(-1, 1)$, it is of the form

$$p(x) = \sum_{j=0}^{d} a_j q_{d,j}(x) \qquad \text{with all} \quad a_j \geqslant 0,$$

where the polynomials $q_{d,j}$ $(0 \leqslant j \leqslant d)$ are defined by (6). For $0 \leqslant b < 1$ we have

$$q'_{d,j}(b) = q_{d,j}(b) \left(\frac{d-j}{1+b} - \frac{j}{1-b} \right) \leqslant d q_{d,j}(b) \qquad (j = 0, 1, ..., d). \tag{7}$$

Multiplying the above inequality by $a_j \geqslant 0$ and adding them up, we obtain $p'(b) \leqslant dp(b)$, which proves the lemma. ∎

Proof of Lemma 3. Without loss of generality we may assume that $0 \leqslant a < 1$. We distinguish two cases.

Case 1. $1 - 2\varepsilon^2 \leqslant a < 1$. Then by Lemma 2 we have

$$d(p) \geqslant \frac{p'(1)}{p(1)} = \frac{2n(1-a)}{(1-a)^2 + \varepsilon^2(1-a^2)} = \frac{2n}{(1-a) + \varepsilon^2(1+a)} > \frac{n}{2\varepsilon^2};$$

thus the lemma is proved.

Case 2. $0 \leqslant a < 1 - 2\varepsilon^2$. First we show that this implies

$$d(p) \geqslant \frac{n}{\varepsilon\sqrt{1-a^2}}. \tag{8}$$

To show this let $b = a + \varepsilon\sqrt{1-a^2} \leqslant 1$. By Lemma 2 we have

$$\begin{aligned} d(p) &\geqslant \frac{p'(b)}{p(b)} = \frac{2n(b-a)}{(b-a)^2 + \varepsilon^2(1-a^2)} = \frac{2n}{\varepsilon\sqrt{1-a^2} + \varepsilon\sqrt{1-a^2}} \\ &= \frac{n}{\varepsilon\sqrt{1-a^2}}, \end{aligned}$$

which proves (8). Further, by Lemma 1 we obtain

$$\left| (1-a^2)\varepsilon^2 - \frac{(1-a^2)n}{4d} \right|^n \leqslant c_4^n \left((1-a^2)\varepsilon^2 + \frac{n^2}{16d^2} \right)^n, \tag{9}$$

where $d = d(p)$ and $n = \frac{1}{2}\deg(p)$. If $(1-a^2)\varepsilon^2 \geqslant (1-a^2)\, n/(8d)$, then there is nothing to prove. Therefore we may assume that

$$(1-a^2)\varepsilon^2 \leqslant \frac{(1-a^2)n}{8d}. \tag{10}$$

Now (9) and (10) yield

$$\left(\frac{(1-a^2)n}{8d}\right)^n \leqslant c_4^n\left((1-a^2)\varepsilon^2+\frac{n^2}{16d^2}\right)^n.$$

Taking the nth root of both sides we obtain

$$\frac{(1-a^2)n}{8d} \leqslant c_4\left((1-a^2)\varepsilon^2+\frac{n^2}{16d^2}\right). \tag{11}$$

From (8) we deduce

$$\frac{n^2}{d^2} \leqslant (1-a^2)\varepsilon^2. \tag{12}$$

Now (11) and (12) imply

$$\frac{(1-a^2)n}{8d} \leqslant c_4 \frac{17}{16}(1-a^2)\varepsilon^2, \tag{13}$$

whence

$$d \geqslant \frac{2}{17} c_4^{-1} \frac{n}{\varepsilon^2} = \frac{c_5 n}{\varepsilon^2};$$

thus the lemma is proved. ∎

5. Proof of Theorems 1 and 2

Proof of Theorem 1. If $\varepsilon \geqslant \frac{1}{2}$, then the obvious inequality $d_n(\varphi) \geqslant n \geqslant n/(4\varepsilon^2)$ gives the desired result. If $0<\varepsilon<\frac{1}{2}$, then let $\{a_j\}_{j=1}^{\infty}$ be such that

$$\frac{\varphi(a_j)}{\sqrt{1-a_j^2}} < \varepsilon_j := \varepsilon + \frac{1}{j+1} \qquad (j=1, 2, ...).$$

This implies that

$$p_j(x) = ((x-a_j)^2 + \varepsilon_j^2(1-a_j^2))^n \in L_{2n}(\varphi);$$

therefore

$$d_{2n}(\varphi) \geqslant \frac{c_5 n}{\varepsilon_j^2} = \frac{c_5 n}{(\varepsilon + 1/(j+1))^2}.$$

Since this holds for every $j = 1, 2, ...$, the lower bound of the theorem is proved. The upper bound follows from Theorem A. ∎

Proof of Theorem 2. Let $q \in \Pi_k \backslash \Pi_{k-1}$ be of the form

$$q(x) = \sum_{j=0}^{k} a_{j,k}(1-x)^j(1+x)^{k-j} \qquad \text{with some real coefficients,} \tag{14}$$

and we define the polynomials

$$q_{\alpha,\beta}(x)=\sum_{j=0}^{k} a_{j,k}(\alpha(1-x))^j(\beta(1+x))^{k-j}. \tag{15}$$

It is easy to see that in case of $\alpha>0$ and $\beta>0$, $q_{\alpha,\beta}$ has no zeros in $(-1, 1)$ if and only if

$$Q(u)=\sum_{j=0}^{k} a_{j,k}u^j \tag{16}$$

has no zeros in $(0, \infty)$. Since $q := q_{1,1}\in \Pi_k\backslash\Pi_{k-1}$ has no zeros in $(-1, 1)$, neither does $q_{\alpha^{-1},\beta^{-1}}$ for every $\alpha>0$ and $\beta>0$. Hence by Bernstein's result [1] (which is a corollary of our Theorem 1), we obtain that

$$q_{\alpha^{-1},\beta^{-1}}(x)\left(\frac{1+x}{2}+\frac{1-x}{2}\right)^m=\sum_{j=0}^{k+m} a_{j,k+m}(1-x)^j(1+x)^{k+m-j}$$

with all $a_{j,k+m}\geqslant 0$ or all $a_{j,k+m}\leqslant 0$ if m is sufficiently large. This yields

$$\begin{aligned}q(x)(\alpha(1-x)+\beta(1+x))^m&=\sum_{j=0}^{k+m} 2^m\alpha^j\beta^{k+m-j}a_{j,k+m}(1-x)^j(1+x)^{k+m-j}\\&=\sum_{j=0}^{k+m} A_{j,k+m}(1-x)^j(1+x)^{k+m-j}\end{aligned} \tag{17}$$

with all $A_{j,k+m}\geqslant 0$ or all $A_{j,k+m}\leqslant 0$ if m is large enough. From (17) we can easily deduce that for every $\alpha>0$, $\beta>0$, and $\gamma>0$

$$\begin{aligned}&q(x)(\alpha(1-x)^2+\beta(1-x)(1+x)+\gamma(1+x)^2)^m\\&\quad=q(x)\left((1-x)\left(\alpha(1-x)+\frac{\beta}{2}(1+x)\right)+(1+x)\left(\frac{\beta}{2}(1-x)+\gamma(1+x)\right)\right)^m\\&\quad=q(x)\sum_{l=0}^{m}\binom{m}{l}(1-x)^l\left(\alpha(1-x)+\frac{\beta}{2}(1+x)\right)^l\\&\qquad\times(1+x)^{m-l}\left(\frac{\beta}{2}(1-x)+\gamma(1+x)\right)^{m-l}\\&\quad=\sum_{j=0}^{k+2m} B_{j,k+2m}(1-x)^j(1+x)^{k+2m-j}\end{aligned} \tag{18}$$

with all $B_{j,k+2m}\geqslant 0$ or all $B_{j,k+2m}\leqslant 0$ if m is large enough. A straightforward calculation shows that $|z|>1$ $(z\in\mathbb{C})$ implies that

$$(x-z)(x-\bar{z})=\alpha(1-x)^2+\beta(1-x)(1+x)+\gamma(1+x)^2$$

with $\alpha = (1 + 2\operatorname{Re} z + |z|^2)/4 > 0$, $\beta = (|z|^2 - 1)/2 > 0$, and $\gamma = (1 - 2\operatorname{Re} z + |z|^2)/4 > 0$; hence (18) shows that $d(q(x)((x-z)(x-\bar{z}))^m) \leqslant k + 2m$ if m is large enough. Since $d(p) \geqslant \deg(p)$ for every polynomial, this yields

$$d(q(x)((x-z)(x-\bar{z}))^m) = k + 2m = \deg(q(x)((x-z)(x-\bar{z}))^m)$$

if m is large enough. Thus the theorem is proved. ∎

6. Markov and Bernstein Type Inequalities for Derivatives of Polynomials from $P_d(-1, 1)$

In 1963 G. G. Lorentz [9] proved the following.

Theorem C. *We have*

$$|p^{(m)}(a)| \leqslant c_6(m)\left(\min\left\{\frac{\sqrt{d}}{\sqrt{1-a^2}}, d\right\}\right)^m \max_{-1 \leqslant x \leqslant 1} |p(x)| \qquad (-1 \leqslant a \leqslant 1)$$

for every polynomial $p \in P_d(-1, 1)$, *where* $c_6(m)$ *depends only on* m.

Relying on Lemma 1 we give a short proof of the above theorem.

Proof of Theorem C. For the sake of brevity, let

$$\Delta_d(a) = \max\left\{\frac{\sqrt{1-a^2}}{\sqrt{d}}, \frac{1}{d}\right\} \qquad (-1 \leqslant a \leqslant 1) \tag{19}$$

and denote by $S_d(a)$ the circle of the complex plane with center a and radius $\frac{1}{4}\Delta_d(a)$. By a slight modification of the proof of Lemma 1, we have

$$|p(z)| \leqslant c_7 \max_{-1 \leqslant x \leqslant 1} |p(x)| \qquad (z \in S_d(a),\ -1 \leqslant a \leqslant 1). \tag{20}$$

Hence by Cauchy's integral formula

$$\begin{aligned} |p^{(m)}(a)| &\leqslant \frac{m!}{2\pi} \int_{S_d(a)} \left|\frac{p(\xi)}{(\xi - a)^{m+1}}\right| |d\xi| \\ &\leqslant \frac{m!}{2\pi} 2\pi \frac{1}{4} \Delta_d(a)(\Delta_d(a))^{-(m+1)} c_7 \max_{-1 \leqslant x \leqslant 1} |p(x)| \\ &\leqslant c_6(m)\left(\min\left\{\frac{\sqrt{d}}{\sqrt{1-a^2}}, d\right\}\right)^m \max_{-1 \leqslant x \leqslant 1} |p(x)| \\ &\qquad (-1 \leqslant a \leqslant 1). \end{aligned} \tag{21}$$

Thus the theorem is proved. ∎

The sharpness of Theorem C was shown in [3] up to the constant $c_6(m)$.

7. Lorentz Degree in the Trigonometric Case

With J. Szabados [5] we introduced the classes $\mathscr{T}_\sigma(\alpha, \beta)$ $(-\pi \leqslant \alpha < \beta < \pi)$ as the family of trigonometric polynomials of the form

$$p(t) = \sum_{j=0}^{2\sigma} a_j \sin^j \frac{\beta - t}{2} \sin^{2\sigma - j} \frac{t - \alpha}{2} \tag{22}$$

with all $a_j \geqslant 0$ or all $a_j \leqslant 0$. For the sake of brevity let $\mathscr{T}_\sigma(\omega) = \mathscr{T}_\sigma(-\omega, \omega)$ $(0 < \omega < \pi)$. For $0 < \omega < \pi/2$ we proved that a trigonometric polynomial $p \not\equiv 0$ is in $\bigcup_{\sigma \in \mathbb{N}} \mathscr{T}_\sigma(\omega)$ if and only if p has no zero in $(-\omega, \omega)$. We also proved that in case of $\pi/2 \leqslant \omega < \pi$ this is not true any more. Similarly to the algebraic case, we introduced the Lorentz degree $\sigma_{[-\omega,\omega]}(p)$ as the smallest natural number σ for which p is of form (22) with $\alpha = -\omega$ and $\beta = \omega$; $\sigma_{[-\omega,\omega]} := \infty$ if there is no such representation. To formulate one of our main results from [5] we need some notations. Let $\varphi(x)$ be a positive continuous function in $(-\omega, \omega)$ and

$$D_\omega(\varphi) = \{z = x + iy \colon |y| < \varphi(x),\ -\omega < x < \omega\}.$$

We introduce

$$L_n(\varphi) = \{p \in T_n \colon p(z) \neq 0 \text{ if } z \in D_\omega(\varphi)\},$$

where T_n is the family of all real trigonometric polynomials of degree at most n. Finally we define

$$\sigma_{n,\omega}(\varphi) = \sup_{p \in L_n(\varphi)} \sigma_{[-\omega,\omega]}(p).$$

In [5] we proved

Theorem D. *Let $0 < \omega < \pi/2$. Then*

$$\sigma_{n,\omega}(\varphi) \leqslant n \left(\frac{4}{\cos\omega} \sup_{|a| < \omega} \frac{\omega^2 - a^2}{\varphi(a)^2} + 2 \tan\omega + 1 \right).$$

On the other hand we have

(i) $\displaystyle \sigma_{n,\omega}(\varphi) \geqslant c_8 n \sup_{|a| < \omega} \frac{(\omega^2 - a^2)^2}{\omega^2 \varphi(a)^2},$

(ii) $\displaystyle \sigma_{n,\omega}(\varphi) \geqslant c_9 n \sup_{|a| < \omega} \frac{(1/\omega) \tan\omega(\omega^2 - a^2)}{\sinh^2 \varphi(a) + (\omega^2 - a^2)(1/\omega^2)}$

and

$$\text{(iii)}\quad \sigma_{n,\omega}(\varphi) \geqslant c_{10} n \sup_{|a|<\omega} \frac{\omega^2 - a^2}{\sinh \varphi(a)(\sinh \varphi(a) + (\omega^2 - a^2)/\omega)}$$

with some absolute constants $c_8 > 0$, $c_9 > 0$, *and* $c_{10} > 0$.

We conjectured the following.

THEOREM 3. *In case of* $0 < \omega < \pi/2$ *we have*

$$c_{11} n \left(\sup_{|a|<\omega} \frac{\omega^2 - a^2}{\varphi(a)^2} + 1 \right) \leqslant \sigma_{n,\omega}(\varphi) \leqslant \frac{c_{12} n}{\cos \omega} \left(\sup_{|a|<\omega} \frac{\omega^2 - a^2}{\varphi(a)^2} + 1 \right), \quad (23)$$

where c_{11} *and* c_{12} *are absolute constants.*

Proof of Theorem 3. The upper bound comes from Theorem D immediately. The lower bound can be obtained by a straightforward modification of the proof of Theorem 1. ▌

Let

$$D = \{z = x + iy \colon -\pi \leqslant x < \pi\}$$

and

$$D_\omega = \{z = x + iy \colon \cos \omega \cosh y < \cos x,\ -\pi \leqslant x < \pi\}.$$

In [5, Theorem 5] we proved

THEOREM E. *Let* $0 < \omega < \pi/2$.

(i) *If* $D_\omega(\varphi) \supseteq D_\omega$, *then*

$$\sigma_{n,\omega}(\varphi) = n \qquad \textit{for all} \quad n = 1, 2, \ldots. \quad (24)$$

(ii) *If* (24) *holds for some* $n \geqslant 1$, *then* $D_\omega(\varphi) \supseteq D_\omega$.

Thus the functions φ for which $\sigma_{n,\omega}(\varphi) = n$ are completely characterized, similarly to the algebraic case. The next theorem shows that a trigonometric polynomial satisfying $\sigma_{[-\omega,\omega]}(p) = \deg(p)$ can have arbitrarily many prescribed zeros outside $(-\omega, \omega)$ if $0 < \omega < \pi/2$.

THEOREM 4. *Let* $0 < \omega < \pi/2$, $z \in D \backslash \bar{D}_\omega$, *and assume that* $q \in T_k \backslash T_{k-1}$ *does not vanish in* $(-\omega, \omega)$. *Then for the trigonometric polynomial* $p(x) = (\sin((x - z)/2) \sin((x - \bar{z})/2))^m q(x)$ *we have* $\sigma_{[-\omega,\omega]}(p) = \deg(p) = k + m$ *if* m *is large enough.*

The above theorem can be proved by a straightforward modification of the proof of Theorem 2, by using the identity

$$1=\left[\frac{1}{\sin^2\omega}\left(\sin^2\frac{\omega-t}{2}+2\cos\omega\sin\frac{\omega-t}{2}\sin\frac{\omega+t}{2}+\sin^2\frac{\omega+t}{2}\right)\right]^m.$$

The condition $0<\omega<\pi/2$ in Theorem 4 is important. If $\pi/2<\omega<\pi$ and $p\in T_k$ has all its zeros in D_ω, then $\sigma_{[-\omega,\omega]}(p)=\infty$. This follows easily from Theorem 2b of [5].

8. Comments

Markov and Bernstein type inequalities for the derivatives of trigonometric polynomials from $\mathcal{T}_\sigma(\omega)$ were established in [6, Theorem 3; 2, Theorem 1; 3, Theorem 1]. Short proofs of some of these results can be given similarly to the proof of Theorem 5. We omit the details.

Related to the lower bound of Theorem 1 the following question arises naturally. Is it true that $d(p)\geqslant cn/\varepsilon^2$ $(0<\varepsilon\leqslant 1)$ for every algebraic polynomial p of degree n having no zeros outside the open ellipse with large axis $(-1,1)$ and small axis $(-\varepsilon i,\varepsilon i)$, where c is an absolute constant? This problem seems to be hard and is open at the moment.

The following example shows that $d(pq)<\max\{d(p),d(q)\}$ can happen. Let $p(x)=(1-x)^2-2(1-x)(1+x)+4(1+x)^2$ and $q(x)=(1+x)+\frac{1}{2}(1-x)$. Then $d(p)=4$, $d(q)=1$, and $d(pq)=3$. Can we estimate $d(pq)$ from below in terms of $d(p)$ and $d(q)$? Maybe $\min\{d(p),d(q)\}$ or $|d(p)-d(q)|$ works. Such lower bounds for $d(pq)$ would be interesting to obtain.

References

1. S. N. Bernstein, Sur la représentation des polynomes positifs, *in* "Collected Papers of S. N. Bernstein," Vol. 1, "Constructive Theory of Functions (1905–1930)," pp. 251–252, Academy of Sciences, USSR, 1952 [in Russian; the original paper was published in 1915].
2. T. Erdélyi, Markov-type inequalities for trigonometric polynomials of special type, *Acta Math. Hungar.* **51**, Nos. 3–4 (1988), 421–436.
3. T. Erdélyi, Markov and Bernstein type inequalities for certain classes of constrained trigonometric polynomials on an interval shorter than the period, *Studia Sci. Math. Hungar.* **25**, Nos. 1–2 (1990), 3–25.
4. T. Erdélyi and J. Szabados, On polynomials with positive coefficients, *J. Approx. Theory* **54** (1988), 107–122.
5. T. Erdélyi and J. Szabados, On trigonometric polynomials with positive coefficients, *Studia Sci. Math. Hungar.* **24**, No. 1 (1989), 71–91.
6. T. Erdélyi and J. Szabados, Bernstein-type inequalities for a class of polynomials, *Acta Math. Hungar.* **53**, Nos. 1–2 (1989), 237–251.

7. B. O. FERGUSON, Approximation by polynomials with integral coefficients, *in* "Math. Surveys No. 17," Amer. Math. Soc., Providence, RI, 1980.
8. G. G. LORENTZ, "Bernstein Polynomials," Toronto Univ. Press, Toronto, 1953.
9. G. G. LORENTZ, The degree of approximation by polynomials with positive coefficients, *Math. Ann.* **151** (1963), 239–251.
10. D. S. LUBINSKY AND Z. ZIEGLER, Coefficient bounds in the Lorentz representation of a polynomial, *Bull. Canad. Math. Soc.*, in press.
11. E. B. SAFF, Incomplete and orthogonal polynomials, *in* "Approximation Theory IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 219–256, Academic Press, New York, 1983.
12. E. B. SAFF AND R. S. VARGA, On incomplete polynomials, *in* "Numerischen Methoden der Approximationstheorie" (L. Collatz, *et al.*, Eds.), Vol. 4, I.S.N.M., Vol. 42, pp. 281–298, Birkhäuser, Basel, 1978.